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Potts models with period doubling cascades, chaos, etc

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Abstract. Various q-state Potts models on Bethe lattices are investigated when q is non-integer. Models with $0 < q < 1$ have been shown to be related to spin glasses as well as polymer models. We find that such systems exhibit period doubling cascades, chaos, etc. Even in the case of a one-dimensional system (a Bethe lattice with branching ratio equal to one) the system has interesting behaviour when $0 < q < 1$ as shown by Glumac and Uzelac through looking at Lee–Yang zeros. We study these systems from a completely different approach to that of dynamical systems.

1. Introduction

The q-state Potts model is one of the most extensively studied models in equilibrium statistical mechanics. For a general review of the Potts model see Wu (1982). The vast majority of the time q is taken to be an integer; however, this is not always the case. In the late 1970s it was pointed out that non-integer q valued Potts models have connections to a number of physical systems of interest, e.g. dilute spin glasses (Aharony and Pfeuty 1979) and gelation and vulcanization of branched polymers (Lubensky and Isaacson 1978). In both cases the connection is made to non-integer q-values between 0 and 1. These connections continue to be of interest today (Whittle 1994).

In this paper we look at q-state Potts models on a Bethe lattice. We study the systems from a dynamical systems approach. A map is derived whose iteration is equivalent to going from a Bethe lattice of n shells to one of $(n + 1)$ shells. In the past this approach has been used for $q = 2$ Potts models by a number of authors (see for example Eggarter 1974, Thompson 1982) as well as general q-state Potts models (see for example Akheyan and Ananikian 1994, deAguiar and Rosa Jr 1992).

In previous studies of q-state Potts models on Bethe lattices there has been little investigation of the situation when q takes on non-integer values. The type of behaviour is crucially dependent on whether the branching ratio, K, of the Bethe lattice under study is equal to one or greater than one. Our results for K greater than one are presented in section 2 along with an introduction to our notation and the Potts–Bethe map. Results for $K = 1$ are presented in section 3. Note that when $K = 1$ one has a one-dimensional system where typically for finite range interactions no phase transition can occur. For $0 < q < 1$ we will show that this is no longer true; phase transitions do occur. This unusual behaviour for $K = 1$ and $0 < q < 1$ has been investigated in a very recent article in this journal (Glumac and Uzelac 1994) from a completely different approach, that of looking at the zeros of the partition function.

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2. Model and notation

For the q-state Potts model with pair interactions the Hamiltonian is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \delta(s_i, s_j) - h \sum_i \delta(s_i, 1) \tag{1}$$

where $s_i = 1, 2, \ldots, q$, where the subscripts i and j denote the ith and jth site. The first sum in (1) is over all nearest neighbour pairs of sites and the second sum is over all sites. The partition function Q is given by

$$Q = \sum_{\{s\}} \exp(-\beta\mathcal{H}) \tag{2}$$

where the sum is over all configurations, which we denote by $\{s\}$, and where $\beta = 1/kT$. The single site 'magnetization' is given by

$$\langle \delta(s_i, 1) \rangle = Q^{-1} \sum_{\{s\}} \delta(s_i, 1) \exp(\beta\mathcal{H}). \tag{3}$$

We are interested in $\langle \delta(s_0, 1) \rangle$, the 'magnetization' of the root site of the Bethe lattice, i.e. the site represented by the open circle in figure 1 which shows a Bethe lattice with a branching ratio, $K = 2$. This root site 'magnetization' can be found by a dynamical systems or recursive approach (Monroe 1991, 1992, 1994, deAguiar *et al* 1991, Akheyan and Ananikian 1994). One considers building up increasingly larger systems of sites by taking K, $(n-1)$th generation trees to form the nth generation tree, as shown for $n = 1$, $n = 2$ and $n = 3$ in figures 1(a), 1(b) and 1(c), respectively. In the limit $n \to \infty$ one has the Bethe lattice. The thermal average of the root site for the nth generation system is given by

$$\langle \delta(s_0, 1) \rangle_n = \frac{a z_n^K}{a z_n^K + (q-1)} \tag{4}$$

where z_n is found from the map

$$z_n = \left[\frac{a b z_{n-1} + (q-1)}{a z_{n-1} + b + (q-2)} \right]^K \tag{5}$$

where $a = \exp[\beta h]$, $b = \exp[\beta J]$, and z_0 is the boundary condition term. Taking $z_0 = 1$ gives the system 'free' boundary conditions. See Monroe (1994), deAguiar *et al* (1991) or Akheyan and Ananikian (1994) for details related to the derivation of equations (4) and (5). The behaviour of $\langle \delta(s_0, 1) \rangle$ is governed by the behaviour of the map, equation (5). As is typical of dynamical systems approaches we want to know when there are fixed points, 2-cycles, etc, for this rational function. As an example for $q = 2$, the Ising spin case, and $J < 0$, the anti-ferromagnetic case, Thompson (1982) showed that for sufficiently low temperatures as one lowered the value of h, with $h > 0$, one found that the map goes from having a single attracting fixed point to having a stable 2-cycle at some value of h which we take to be h_1, that is, a period doubling bifurcation. If one continues to lower the value of h to $h < 0$ when one reaches h_2 with $h_2 = -h_1$ the stable 2-cycle becomes unstable and one returns to the case where there is a single attracting fixed point. For q integer and $q > 2$ Akheyan and Ananikian (1994) found a similar behaviour except $h_1 \neq h_2$.

Akheyan and Ananikian (1994) remark that only one period doubling occurs in the anti-ferromagnetic Potts model on the Bethe lattice and that only by adding a next-nearest-neighbour interaction in the Hamiltonian does one obtain the full bifurcation diagram with period-three windows, chaos, etc. This, we point out, is only true if one restricts oneself to non-negative integer q values. For systems with non-integer q values in the range

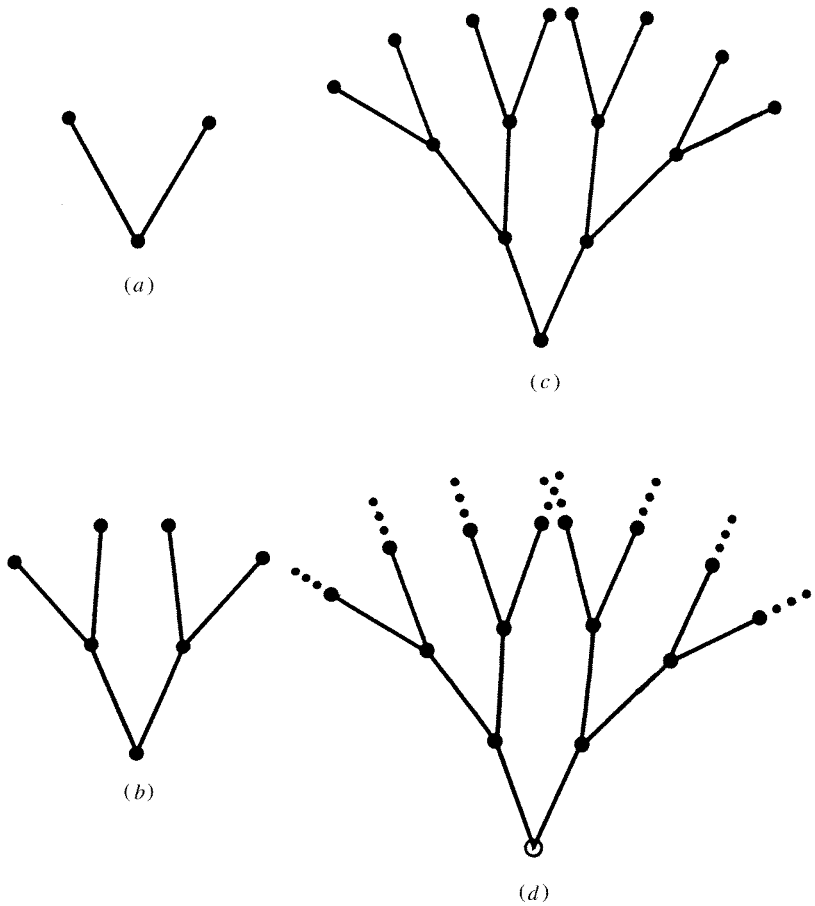

Figure 1. Steps in the construction of the Bethe lattice with $K = 2$. (a) First generation branch, (b) second generation branch, (c) third generation branch and (d) completion of the Bethe lattice.

$0 < q < 2$ and with anti-ferromagnetic interactions or for systems with non-integer q values in the range $0 < q < 1$ and with ferromagnetic nearest neighbour interactions one obtains for $\langle \delta(s_0, 1) \rangle$ versus h bifurcation diagrams with the full range of characteristics typical of bifurcation diagrams for dynamical systems including the prototypical system $z^2 + c$. We illustrate this for the anti-ferromagnetic case in figure 2 with several examples of plots of $\langle \delta(s_0, 1) \rangle$ versus h and in figure 3 with several examples involving ferromagnetic interactions.

3. $0 < q < 1$ and $K = 1$

In this section we confine our attention to one-dimensional systems. A Bethe lattice with branching ratio equal to one is simply a one-dimensional system. In a very recent article in this journal Glumac and Uzelac (1994) have studied the zeros of the partition function of one-dimensional q-state Potts models including cases where q is non-integer. They point out the very different behaviour of the zeros for $q > 1$ as compared to $q < 1$. For $q < 1$

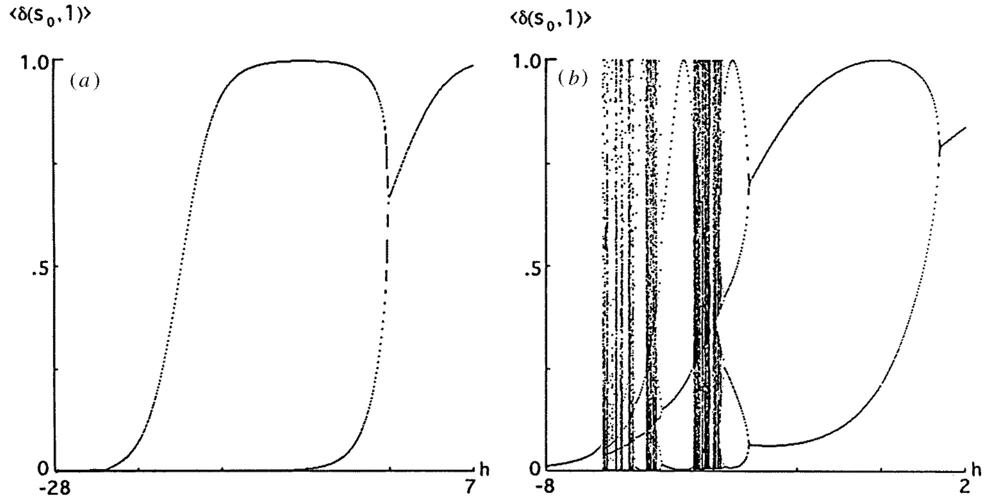

Figure 2. Plots of $\langle\delta(s_0, 1)\rangle$ versus h with $q = 1.5$, $K = 2$, $J = -1$, and in (a) $kT = 1.45$ and in (b) $kT = 1.00$.

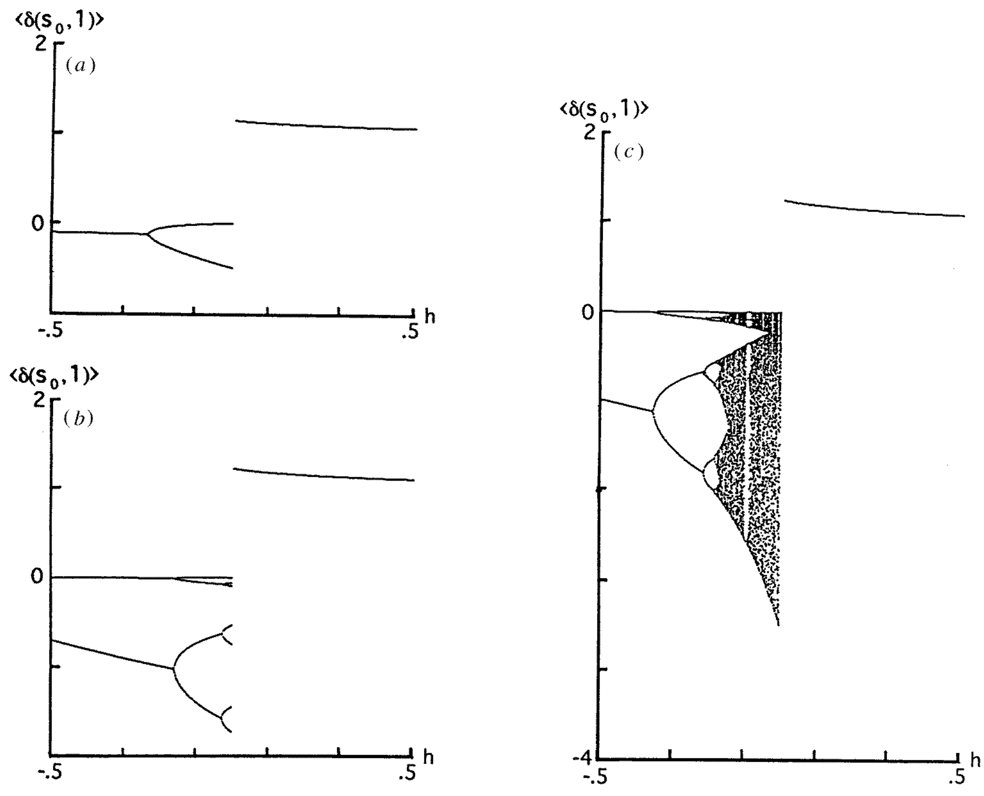

Figure 3. Plots of $\langle\delta(s_0, 1)\rangle$ versus h with $q = 0.5$, $K = 2$, $J = 1$ and in (a) $kT = 1.00$, (b) $kT = 1.15$ and (c) $kT = 1.20$.

the zeros lie on an interval of the positive real z-axis. Hence contrary to the usual statement that one does not have phase transitions in one-dimensional lattice spin systems, one may have them in these systems.

We continue to use the dynamical system approach and take $K = 1$. Then the map, equation (5), is a Möbius transformation or linear fractional transformation (Ahlfors 1966). The dynamics of such maps is rather simple and well understood (Beardon 1991). For specificity we begin by looking at the case of $q = \frac{1}{2}$. The map is

$$z_n = \frac{abz_{n-1} - \frac{1}{2}}{az_{n-1} + b - \frac{3}{2}}. \tag{6}$$

We point out that one can find the fixed points analytically and there are two of them because in the case of $K = 1$ we are only dealing with a quadratic equation when we require $z_n = z_{n-1}$. We now restrict ourselves still further by taking $kT = J = 1$. For such a map the behaviour of z is such that for $a > a_1$ and $a < a_2$ there are two real-valued fixed points, the larger (smaller) one being attracting (repelling). For $a_2 < a < a_1$ there are two complex-valued fixed points, both of which are indifferent or neutral fixed points. For the latter case, as shown in Beardon (1991), one has either that the z_n are dense on a circle of the complex z-plane or they are periodic on a circle in the complex z-plane. Thus there is a phase transition at h_1 where $a_1 = \exp(\beta h_1)$ and at h_2 where $a_2 = \exp(\beta h_2)$. The special points a_1 and a_2 are easily calculated as this is when the two fixed points are equal to each other. Our results confirm those of Glumac and Uzelac (1994) using this very different approach. They find using their notation that the zeros of the partition function lie on an interval of the real z-axis when $z_- \leqslant z \leqslant z_+$. Their z_- and z_+ are exactly our a_1 and a_2.

If we relax the condition that $J = 1$ then we see that for all $J > 0$ there are values a_1 and a_2 between which the behaviour as described above is found. In figure 4(b) we have a plot of $\exp(\beta h)$ versus βJ. The cross-hatched regions indicate where one has z_n either dense on a circle in the complex z-plane or periodic on such a circle. For $J > 0$ the cross-hatched area corresponds exactly to the area between the broken and dotted curves of Glumac and Uzelac's figure 2. If one has $J < 0$ then again one has a shaded region with

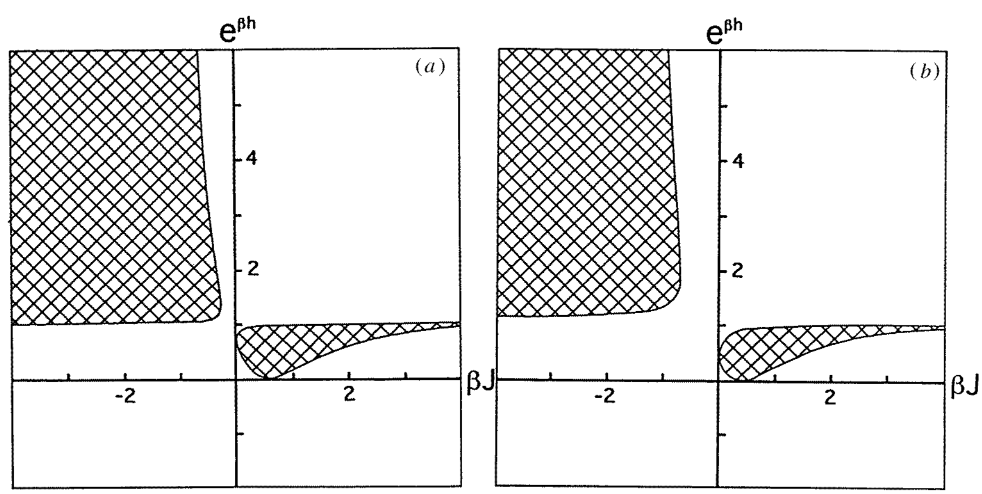

Figure 4. Plots of $\exp(\beta h)$ versus βJ where in the cross-hatched regions one has z_n either dense on a circle in the complex z-plane or periodic on such a circle and where in the non-cross-hatched region z_n for $n \to \infty$ goes to an attracting fixed point. In (a) $q = 0.25$ and in (b) $q = 0.50$.

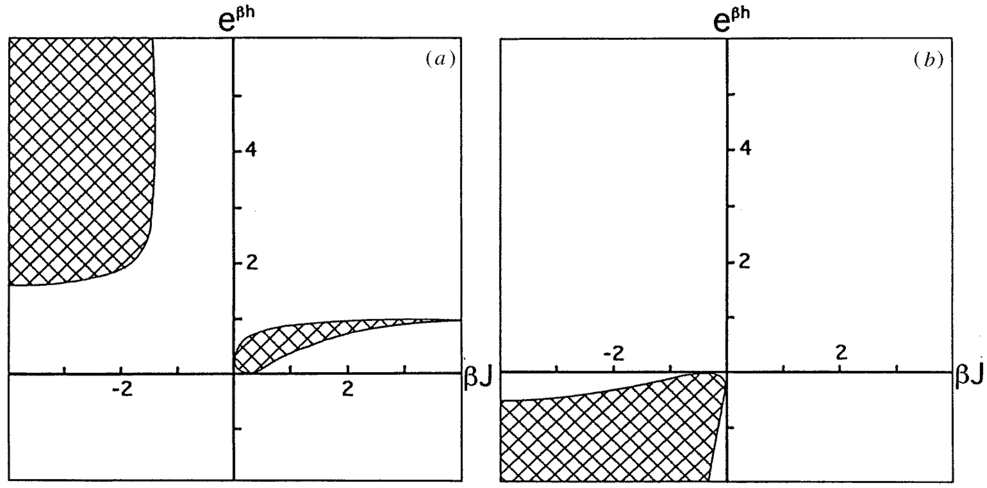

Figure 5. Plots of $\exp(\beta h)$ versus βJ where in the cross-hatched regions one has z_n either dense on a circle in the complex z-plane or periodic on such a circle and where in the non-cross-hatched region z_n for $n \to \infty$ goes to an attracting fixed point. In (a) $q = 0.75$ and in (b) $q = 1.25$.

the behaviour described above, only here this is not true for all $J < 0$. Rather for $q = \frac{1}{2}$ and $kT = 1$ we need $J < -0.6931\ldots$. Figures 4(a), 5(a) and 5(b) show the situation for $q = \frac{1}{4}$, $q = \frac{3}{4}$, and $q = \frac{5}{4}$, respectively. For $q > 1$, as exemplified by figure 5(b), there is no shaded area for h real and hence no phase transition occurs. One can see this by finding the fixed points of equation (5) when $K = 1$ and then finding values of a and b such that the two fixed points are equal. When one does this one sees that if $q > 1$ the values of a and b are either complex or negative, both of which have no physical significance.

4. Conclusions

We have shown that it is not necessary to go to next-nearest-neighbour interactions to obtain the full characteristics, period doubling cascades, etc, of bifurcation diagrams when one plots $\langle\delta(s_0, 1)\rangle$ versus h. Rather, these characteristics are exhibited in plots of $\langle\delta(s_0, 1)\rangle$ versus h when one considers non-integer q values. This can occur even in the case of a Bethe lattice with branching ratio $K = 1$ which is nothing more than a one-dimensional lattice system. For the $K = 1$ case our results complement the very recent results of Glumac and Uzelac (1994) who investigated the case of non-integer q value Potts model systems in one dimension using the Lee–Yang zeros approach. The general period doubling scheme accompanied by chaotic behaviour has been shown to occur in systems with frustration and is associated with spin glass behaviour, see e.g. McKay *et al* (1982). Finally, it would be remiss of us if we did not point out that a number of authors have emphasized some of the pathologies of non-integer q-state Potts model systems, e.g. Griffiths and Gujrati (1983).

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